Appendix: The Laplace Transform

The **Laplace transform** is a powerful method that can be used to solve differential equation, and other mathematical problems. Its strength lies in the fact that it allows the transformation of a *differential* equation to an *algebraic* equation.

The *one-sided* Laplace transform is defined as follows

$$X(s) \equiv L\left[x(t)\right] = \int_0^\infty x(t)\,e^{-st}\,dt, \tag{I.1}$$

where the variable s is defined as containing both a real and imaginary part, i.e., $s = \sigma + i\omega$ with $\sigma \geq 0$ such that e^{-st} remains finite as $t \to \infty$. Referring to equation (I.1) we say that "$X(s)$ is the Laplace transform of $x(t)$". We also assumed that the time variable t starts at 0, but this could be changed to any other value (e.g., t_0).

For example, we calculate the Laplace transform of a few simple functions

$$\begin{aligned} L[A] &= \int_0^\infty A e^{-st}\,dt = -\frac{A}{s}e^{-st}\Big|_0^\infty = \frac{A}{s}, && s > 0 \\ L\left[e^{-at}\right] &= \int_0^\infty e^{-at}e^{-st}\,dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}, && s > -a. \end{aligned} \tag{I.2}$$

A particularly important transform is that of an impulse of time duration τ defined as

$$\begin{aligned} x(t) &= \frac{1}{\tau}, && 0 < t < \tau \\ &= 0, && |t| > \tau \end{aligned} \tag{I.3}$$

with

$$X(s) = \frac{1}{\tau}\int_0^\tau e^{-st}\,dt = -\frac{1}{s\tau}e^{-st}\Big|_0^\tau = \frac{1}{s\tau}\left(1 - e^{-s\tau}\right). \tag{I.4}$$

If we now take the limit of equation (I.4) when $\tau \to 0$, we get

$$\lim_{\tau \to 0} X(s) = \frac{1}{s\tau}\left(1 - \left[1 - s\tau\right]\right) = 1. \tag{I.5}$$

The limit of a function such as $x(t)$, defined in equation (I.3), when the duration of the impulse is taken to be infinitely small while keeping the area of the impulse constant is called a **Dirac** or **delta function**. It is usually simply written as $\delta(t)$, and has the property that $\delta(t)=0$ for $t \neq 0$ and $\delta(t)=\infty$ for $t=0$, but

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1. \tag{I.6}$$

Equally important is the transform of a **step** or **Heaviside function**, represented by

$$\begin{aligned} H(t) &= 1, \qquad t>0 \\ &= 0, \qquad t<0. \end{aligned} \tag{I.7}$$

Since our definition of the Laplace transform truncates any functions that are non-zero for $t<0$, the Laplace transform of the step function was evaluated in equation (I.2) and found to be

$$L\left[H(t)\right] = \frac{1}{s}. \tag{I.8}$$

The list of transforms appearing in Table I.1 can be similarly verified.

Table I.1 – Laplace transform pairs

$$L\left[A\delta(t)\right] = A, \qquad s>0$$

$$L\left[AH(t)\right] = \frac{A}{s}, \qquad s>0$$

$$L\left[e^{-at}H(t)\right] = \frac{1}{s+a}, \qquad s>-a$$

$$L\left[t^n H(t)\right] = \frac{n!}{s^{n+1}}, \qquad s>0, \quad n=1,2,3,\ldots$$

$$L\left[t^n e^{-at} H(t)\right] = \frac{n!}{(s+a)^{n+1}}, \qquad s>-a, \quad n=1,2,3,\ldots$$

$$L\left[\sin(\omega t)H(t)\right] = \frac{\omega}{s^2+\omega^2}, \qquad s>0$$

$$L\left[\cos(\omega t)H(t)\right] = \frac{s}{s^2+\omega^2}, \qquad s>0$$

$$L\left[e^{-at}\sin(\omega t)H(t)\right] = \frac{\omega}{(s+a)^2+\omega^2}, \qquad s>-a$$

$$L\left[e^{-at}\cos(\omega t)H(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}, \qquad s>-a.$$

The Laplace transform also possesses other important properties, some of which are (assuming that every time function is zero for $t<0$ in what follows)

I. Linearity. If A and B are constants

$$L\left[Ax(t)+By(t)\right]=AX(s)+BY(s) \tag{I.9}$$

II. Transform of derivatives.

$$\begin{aligned} L\left[\frac{dx(t)}{dt}\right] &= \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt = x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt \\ &= sX(s)-x(0) \end{aligned} \tag{I.10}$$

where we integrated by parts, and $x(0)$ is the initial condition $x(t)$. Similarly, the transform of higher derivatives can be shown to give

$$\begin{aligned} L\left[\frac{d^2x(t)}{dt^2}\right] &= s^2X(s)-sx(0)-\frac{dx(t)}{dt}\bigg|_{t=0} \\ L\left[\frac{d^nx(t)}{dt^n}\right] &= s^nX(s)-\sum_{k=1}^{n}s^{n-k}\frac{d^{k-1}x(t)}{dt^{k-1}}\bigg|_{t=0} \end{aligned} \tag{I.11}$$

III. Transform of primitive of functions.

$$\begin{aligned} L\left[\int_0^t x(\tau)d\tau\right] &= \int_0^\infty\left\{\int_0^t x(\tau)d\tau\right\}e^{-st}\,dt \\ &= -\frac{1}{s}\left\{\int_0^t x(\tau)d\tau\right\}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty x(t)e^{-st}\,dt \\ &= \frac{X(s)}{s}+\frac{1}{s}\left\{\int_0^t x(\tau)d\tau\right\}\bigg|_{t=0} \end{aligned} \tag{I.12}$$

where we again integrated by parts. The transform of higher primitives is given by

$$L\left[\int\ldots\int x(\tau)(d\tau)^n\right]=\frac{X(s)}{s^n}+\sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left\{\int\ldots\int x(\tau)(d\tau)^k\right\}\bigg|_{t=0} \tag{I.13}$$

IV. Time shifting. Since $x(t)=0$ for $t<0$, we can write

$$\begin{aligned} L\left[x(t-\tau)\right] &= \int_0^\infty x(t-\tau)e^{-st}dt = \int_\tau^\infty x(t-\tau)e^{-st}dt \\ &= \int_0^\infty x(\lambda)e^{-s(\lambda+\tau)}d\lambda = e^{-s\tau}\int_0^\infty x(\lambda)e^{-s\lambda}d\lambda \\ &= e^{-s\tau}X(s) \end{aligned} \tag{I.14}$$

where we made the substitution $\lambda = t-\tau$.

V. Multiplication by an exponential.

$$\begin{aligned} L\left[e^{-at}x(t)\right] &= \int_0^\infty e^{-at}x(t)e^{-st}dt = \int_0^\infty x(t)e^{-(s+a)t}dt \\ &= X(s+a) \end{aligned} \tag{I.15}$$

The residue theorem

Once a function or an equation has been transformed in the Laplace domain, then modified for one purpose or another, it will eventually need to be transformed back to the time domain. Although an inverse Laplace transform can be mathematically defined, it is always more convenient and easier to use the so-called *residue theorem* to go from the Laplace to the time domain. This theorem is stated as follows. Given a function $X(s)$, for which the denominator can be written as a product of factors of the type $\left(s+a_j\right)^m$ (where a_j is called a *pole* of order *m*), we can write

$$\begin{aligned} x(t) &= L^{-1}\left[X(s)\right] \\ &= \sum_{j=1}^{n}\frac{1}{(m-1)!}\lim_{s\to -a_j}\left(\frac{d^{m-1}}{ds^{m-1}}\left[\left(s+a_j\right)^m X(s)e^{st}\right]\right), \quad t>0 \end{aligned} \tag{I.16}$$

where *n* is the number of poles in the denominator of $X(s)$, and the quantity in between the curly braces is called the residue of $X(s)e^{st}$ at the pole a_j of order *m*. Let's consider a few examples

$$x(t) = L^{-1}\left[\frac{1}{s+a}\right] = \lim_{s\to -a}\frac{1}{0!}\frac{d^0}{ds^0}\left[(s+a)\cdot\frac{e^{st}}{s+a}\right]$$
$$= e^{-at}, \qquad t > 0$$
$$x(t) = L^{-1}\left[\frac{1}{(s+a)^2}\right] \tag{I.17}$$
$$= \lim_{s\to -a}\frac{1}{1!}\frac{d^1}{ds^1}\left[(s+a)^2\cdot\frac{e^{st}}{(s+a)^2}\right]$$
$$= te^{-at}, \qquad t > 0$$

and finally

$$x(t) = L^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = L^{-1}\left[\frac{s+a}{\left(s+(a-i\omega)\right)\left(s+(a+i\omega)\right)}\right]$$
$$= \lim_{s\to -a+i\omega}\frac{1}{0!}\frac{d^0}{ds^0}\left[\left(s+(a-i\omega)\right)\frac{(s+a)}{(s+a)^2+\omega^2}e^{st}\right]$$
$$+ \lim_{s\to -a-i\omega}\frac{1}{0!}\frac{d^0}{ds^0}\left[\left(s+(a+i\omega)\right)\frac{(s+a)}{(s+a)^2+\omega^2}e^{st}\right] \tag{I.18}$$
$$= \frac{i\omega e^{(-a+i\omega)t}}{2i\omega} + \frac{-i\omega e^{(-a-i\omega)t}}{-2i\omega}$$
$$= e^{-at}\cos(\omega t), \qquad t > 0.$$

These results can be verified against the examples presented in Table I.1.

Application to the damped oscillator problem

Let's now solve a few cases involving the equation of motion of a damped oscillator with different types of driving input. The equation to solve is

$$\boxed{\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^{\,2}x(t) = f(t)} \tag{I.19}$$

I. $f(t) = A\delta(t)$.

$$L\left[\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^{\,2}x(t)\right] = L\left[f(t)\right] \tag{I.20}$$

Using the linearity property of the Laplace transform and Table I.1, we get

$$\left(s^2X(s) - sx_0 - \dot{x}_0\right) + 2\beta\left(sX(s) - x_0\right) + \omega_0^{\,2}X(s) = A, \tag{I.21}$$

or

$$X(s)\left(s^2+2\beta s+\omega_0^{\ 2}\right)=A+x_0\left(s+2\beta\right)+\dot{x}_0. \tag{I.22}$$

In everything that will follow, we will assume that $x_0=\dot{x}_0=0$. We now solve equation (I.21)

$$\begin{aligned} X(s) &= \frac{A}{s^2+2\beta s+\omega_0^{\ 2}} \\ &= \frac{A}{\left(s+\left(\beta-\sqrt{\beta^2-\omega_0^{\ 2}}\right)\right)\left(s+\left(\beta+\sqrt{\beta^2-\omega_0^{\ 2}}\right)\right)} \end{aligned} \tag{I.23}$$

We now use the residue theorem stated in equation (I.16)

$$\begin{aligned} x(t) &= A\left[\frac{e^{-\left(\beta-\sqrt{\beta^2-\omega_0^{\ 2}}\right)t}}{2\sqrt{\beta^2-\omega_0^{\ 2}}}-\frac{e^{-\left(\beta+\sqrt{\beta^2-\omega_0^{\ 2}}\right)t}}{2\sqrt{\beta^2-\omega_0^{\ 2}}}\right] \\ &= A\frac{e^{-\beta t}}{2\sqrt{\beta^2-\omega_0^{\ 2}}}\left[e^{\sqrt{\beta^2-\omega_0^{\ 2}}\,t}-e^{-\sqrt{\beta^2-\omega_0^{\ 2}}\,t}\right], \qquad t>0 \end{aligned} \tag{I.24}$$

A close examination of equation (I.24) shows that the response of the damped oscillator to a Dirac function is nothing more than the complementary solution of the equation of motion. In the case of the underdamped oscillator ($\beta^2<\omega_0^{\ 2}$), we find that

$$x(t)=A\frac{e^{-\beta t}}{\omega_1}\sin\left(\omega_1 t\right), \qquad t>0 \tag{I.25}$$

with $\omega_1=\sqrt{\omega_0^{\ 2}-\beta^2}$.

II. $f(t)=AH(t)$

In this case, we have (assuming that $\beta^2<\omega_0^{\ 2}$, and $\omega_1=\sqrt{\omega_0^{\ 2}-\beta^2}$)

$$X(s)\left(s^2+2\beta s+\omega_0^{\ 2}\right)=\frac{A}{s} \tag{I.26}$$

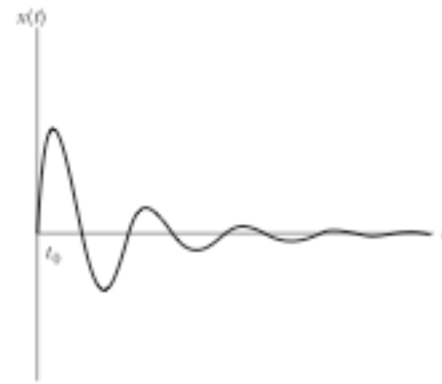

Figure I.1 – Response to a Dirac function driving input.

$$
\begin{aligned}
x(t) &= \frac{A}{s\left(s^2+2\beta s+{\omega_0}^2\right)} \\
&= \frac{A}{s\left(s+\left(\beta-i\omega_1\right)\right)\left(s+\left(\beta+i\omega_1\right)\right)} \\
&= A\left[\frac{1}{{\omega_0}^2}+\frac{e^{-(\beta-i\omega_1)t}}{2i\omega_1\left(i\omega_1-\beta\right)}+\frac{e^{-(\beta+i\omega_1)t}}{2i\omega_1\left(i\omega_1+\beta\right)}\right] \qquad \text{(I.27)} \\
&= A\left[\frac{1}{{\omega_0}^2}-\frac{e^{-\beta t}}{\omega_1\sqrt{\beta^2+{\omega_1}^2}}\cos\left(\omega_1 t-\phi\right)\right], \qquad t>0
\end{aligned}
$$

with

$$\phi = \tan^{-1}\left(\frac{\beta}{\omega_1}\right). \qquad \text{(I.28)}$$

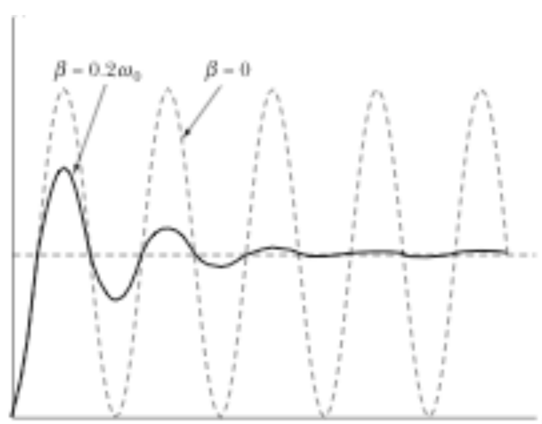

Figure I.2 – Response to a step function as driving input.

The Laplace transform can be systematically applied to more complicated types of problems and driving functions (periodic or not). It is also important to realize that the solution to a given problem provided by the application of the Laplace transform **includes both the complementary and the particular solutions**.

The Two-sided Laplace Transform

It is generally the case in physics that a function is not limited to $t \geq 0$ but can exist for times both positive and negative. We can then generalize the one-sided Laplace transform given in equation (I.1) with its *two-sided* version

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt. \tag{I.29}$$

Using our previous notation for the one-sided transform

$$L[x(t)](s) = \int_{0}^{\infty} x(t) e^{-st} \, dt \tag{I.30}$$

we can write for the two-sided transform

$$\begin{aligned} X(s) &= \int_{0}^{\infty} x(t) e^{-st} \, dt + \int_{-\infty}^{0} x(t) e^{-st} \, dt \\ &= \underbrace{L[x(t)](s)}_{t>0} + \underbrace{L[x(-t)](-s)}_{t<0}. \end{aligned} \tag{I.31}$$

In equation (I.30) we made the dependency of the Laplace transform on the parameter s explicit by adding '(s)' to the left-hand side. The last term on the right-hand side of the second of equations (I.31) can be ascertained with (using $\lambda = -t$)

$$\begin{aligned} L[x(-t)](-s) &= \int_{0}^{\infty} x(-t) e^{st} \, dt \\ &= -\int_{0}^{-\infty} x(\lambda) e^{-s\lambda} \, d\lambda \\ &= \int_{-\infty}^{0} x(\lambda) e^{-s\lambda} \, d\lambda. \end{aligned} \tag{I.32}$$

For example, if we calculate the two-sided Laplace transform of the following function

$$x(t) = \begin{cases} -e^{2t}, & t < 0 \\ -e^{-3t}, & t \geq 0 \end{cases} \tag{I.33}$$

we find (using Table I.1)

$$
\begin{aligned}
X(s) &= -\underbrace{\frac{1}{-s+2}}_{s<2} - \underbrace{\frac{1}{s+3}}_{s>-3} \\
&= \frac{5}{s^2+s-5}, \quad \text{for } -3<s<2.
\end{aligned}
\tag{I.34}
$$

We must also determine the proper relation to calculate inverse Laplace transforms for two-sided functions. Using equation (I.32) and the residue theorem for one-sided functions (i.e., equation (I.16)) we can write

$$
\begin{aligned}
x(-t) &= L^{-1}\left[X(-s)\right] \\
&= \sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to -a_j} \left(\frac{d^{m-1}}{ds^{m-1}} \left[\left(s+a_j\right)^m X(-s) e^{st} \right] \right), \quad t>0.
\end{aligned}
\tag{I.35}
$$

We can now look at negative time values by changing $t \to -t$, and

$$
x(t) = \sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to -a_j} \left(\frac{d^{m-1}}{ds^{m-1}} \left[\left(s+a_j\right)^m X(-s) e^{-st} \right] \right), \quad t<0.
\tag{I.36}
$$

We finally also change $s \to -s$ to find

$$
\begin{aligned}
x(t) &= \sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to a_j} \left(\frac{d^{m-1}}{d(-s)^{m-1}} \left[\left(-s+a_j\right)^m X(s) e^{st} \right] \right) \\
&= \sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to a_j} \left(\frac{1}{(-1)^{m-1}} \frac{d^{m-1}}{ds^{m-1}} \left[(-1)^m \left(s-a_j\right)^m X(s) e^{st} \right] \right) \\
&= -\sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to a_j} \left(\frac{d^{m-1}}{ds^{m-1}} \left[\left(s-a_j\right)^m X(s) e^{st} \right] \right), \quad t<0.
\end{aligned}
\tag{I.37}
$$

That is, the inverse Laplace transform for a function defined only for times $t<0$ is similar to that for a function defined for $t>0$, except for the overall negative sign. For example, if we calculate the inverse Laplace transform of equation (I.34) we find

$$
\begin{aligned}
x(t) &= -\frac{5(s-2)}{s^2+s-5} e^{2t} H(-t) + \frac{5(s+3)}{s^2+s-5} e^{-3t} H(t) \\
&= -e^{2t} H(-t) - e^{-3t} H(t),
\end{aligned}
\tag{I.38}
$$

which is the same as equation (I.33).

Finally, we note that because the two-sided Laplace transform consists of an integral performed over the domain $-\infty < t < \infty$ the dependencies on derivatives and integrals evaluated at $t=0$ (see equations (I.11) and (I.13)) do not appear in the results of calculations. Notably we have

$$\begin{gathered}
\int_{-\infty}^{\infty}\left[\frac{dx(t)}{dt}\right]^{n}e^{-st}dt=s^{n}X(s)\\
\int_{-\infty}^{\infty}\left\{\int\dots\int x(\tau)(d\tau)^{n}\right\}e^{-st}\,dt=\frac{X(s)}{s^{n}}.
\end{gathered} \tag{I.39}$$